

# Buffon's problem with a pivot needle

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## Abstract

In this paper, we solve Buffon's needle problem for a needle consisting of two line segments connected in a pivot point.

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## 1 Introduction

The classical Buffon needle problem asks for the probability that a needle of length  $\ell$  thrown at random onto a plane lattice  $\mathcal{R}_d$  of parallel lines at a distance  $d \geq \ell$  apart will hit one of these lines. This problem was stated and solved by Buffon in his *Essai d'Arithmétique Morale*, 1777 (see e. g. [5, pp. 71-72], [6, pp. 501-502]). If an arbitrary convex body  $\mathcal{C}$  with maximum width  $\leq d$  is used in this experiment, then the hitting probability is given by  $u/(\pi d)$ , where  $u$  denotes the perimeter of  $\mathcal{C}$ . This is the result of Barbier in 1860 [1, pp. 274-275], [6, p. 507]. If  $\mathcal{C}$  is a needle (line segment), then  $u = 2\ell$ . If  $\mathcal{C}$  is an ellipse, then there are elliptic integrals in the formulas of the hitting probabilities, see Duma and Stoka [3].

We consider a needle  $\mathcal{N}_{a,b}$  consisting of two line segments  $C'A'$ ,  $C'B'$  of lengths  $a := |C'A'|$  and  $b := |C'B'|$ , connected in a pivot point  $C'$  (see Fig. 1), and assume  $a + b \leq d$ . The *random throw of  $\mathcal{N}_{a,b}$  onto  $\mathcal{R}_d$*  is defined as follows: The  $y$ -coordinate of the point  $C'$  is a random variable uniformly distributed in  $[0, d]$ . The angles  $\alpha$  and  $\beta$  between the lines of  $\mathcal{R}_d$ , and segments  $C'A'$  and  $C'B'$ , respectively, are random variables uniformly distributed in  $[0, 2\pi]$ . All three random variables are stochastically independent.

The probability of the event that  $\mathcal{N}_{a,b}$  hits two lines of  $\mathcal{R}_d$  at the same time is equal to zero, even in the case  $a + b = d$ . The expectation  $\mathbb{E}(n)$  of the random variable  $n = \text{number of intersection points between } \mathcal{N}_{a,b} \text{ and } \mathcal{R}_d$  is given by  $\mathbb{E}(n) = 2(a + b)/(\pi d)$ , cp. [4].

Here we are asking for the probabilities  $p(i)$ ,  $i \in \{0, 1, 2\}$ , of the events that  $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  in exactly  $i$  points. We denote by  $A$  and  $B$  the events that segments  $C'A'$  and  $C'B'$ , respectively, hit one line of  $\mathcal{R}_d$ .

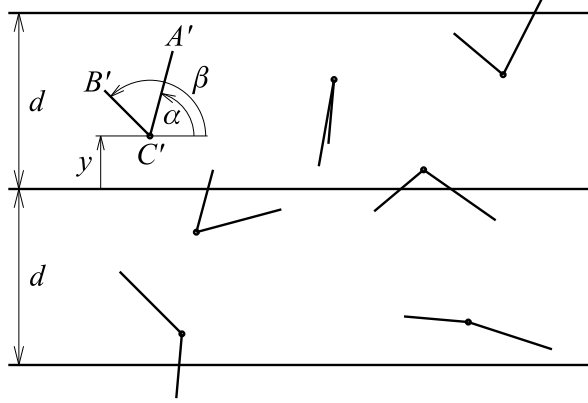


Fig. 1: Lattice  $\mathcal{R}_d$  and randomly thrown needle  $\mathcal{N}_{a,b}$

## 2 Hitting probabilities

**Theorem.** *If  $a + b \leq d$ , then the probabilities  $p(i)$  that  $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  in exactly  $i$  points are given by*

$$p(0) = 1 - \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \quad p(1) = \frac{4(a+b)E(k)}{\pi^2 d},$$

$$p(2) = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d},$$

where

$$E(k) = E(\pi/2, k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

is the complete elliptic integral of the second kind with  $k^2 = 4ab/(a+b)^2$ .

*Proof.* We observe that the angle  $\phi := \angle(C'A', C'B')$  is a random variable uniformly distributed in  $[0, 2\pi]$ . Due to the result of Barbier, the conditional probability  $P(A \cup B | \phi)$  of  $A \cup B$  for fixed value of  $\phi \in [0, 2\pi]$  is given by  $u(\phi)/(\pi d)$ , where  $u(\phi)$  is the perimeter of the convex hull of  $\mathcal{N}_{a,b}$ . ( $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  if and only if its convex hull hits  $\mathcal{R}_d$ .) Using the law of total probability, the probability that  $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  is given by

$$P(A \cup B) = \int_0^{2\pi} P(A \cup B | \phi) \frac{d\phi}{2\pi} = \frac{1}{2\pi^2 d} \int_0^{2\pi} u(\phi) \, d\phi$$

$$= \frac{1}{2\pi^2 d} \int_0^{2\pi} [a + b + c(\phi)] \, d\phi = \frac{a + b + \bar{c}}{\pi d},$$

where  $c := |A'B'|$ , and

$$\bar{c} := \frac{1}{2\pi} \int_0^{2\pi} c(\phi) \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a^2 + b^2 - 2ab \cos \phi} \, d\phi.$$

Using  $\cos \phi = 2 \cos^2(\phi/2) - 1$ , we have

$$\begin{aligned}\bar{c} &= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(a+b)^2 - 4ab \cos^2 \frac{\phi}{2}} \, d\phi \\ &= \frac{a+b}{2\pi} \int_0^{2\pi} \sqrt{1 - \frac{4ab}{(a+b)^2} \cos^2 \frac{\phi}{2}} \, d\phi.\end{aligned}$$

For abbreviation we put  $k^2 = 4ab/(a+b)^2$ . From the inequality  $\sqrt{ab} \leq (a+b)/2$  between the geometric and the arithmetic mean, one finds  $k^2 \leq 1$ , hence  $0 \leq k \leq 1$  with  $k = 1$  only for  $a = b$ . With the substitution  $\chi = \phi/2$  we get

$$\begin{aligned}\bar{c} &= \frac{a+b}{\pi} \int_0^\pi \sqrt{1 - k^2 \cos^2 \chi} \, d\chi = \frac{2(a+b)}{\pi} \int_0^{\pi/2} \sqrt{1 - k^2 \cos^2 \chi} \, d\chi \\ &= \frac{2(a+b)}{\pi} \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \chi} \, d\chi = \frac{2(a+b)E(k)}{\pi}.\end{aligned}$$

It follows that

$$\begin{aligned}P(A \cup B) &= \frac{a+b+\bar{c}}{\pi d} = \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \\ P(A \cap B) &= P(A) + P(B) - P(A \cup B) = \frac{2a}{\pi d} + \frac{2b}{\pi d} - \frac{a+b+\bar{c}}{\pi d} \\ &= \frac{a+b-\bar{c}}{\pi d} = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d},\end{aligned}$$

and

$$\begin{aligned}p(0) &= 1 - P(A \cup B) = 1 - \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \\ p(1) &= P(A \cup B) - P(A \cap B) = \frac{a+b+\bar{c}}{\pi d} - \frac{a+b-\bar{c}}{\pi d} = \frac{2\bar{c}}{\pi d} \\ &= \frac{4(a+b)E(k)}{\pi^2 d}, \\ p(2) &= P(A \cap B) = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d}.\end{aligned}\quad \square$$

This is the result from [2, pp. 57-58]. There it was obtained as special case of the more general result in Corollary 4.2 [2, p. 56].

**Remark.** If the angle  $\phi$  is constant, then we have

$$P(A \cup B) = \frac{a+b+c}{\pi d} \quad \text{and} \quad P(A \cap B) = \frac{a+b-c}{\pi d}$$

with  $c = \sqrt{a^2 + b^2 - 2ab \cos \phi}$ . This yields

$$p(0) = 1 - \frac{a+b+c}{\pi d}, \quad p(1) = \frac{2c}{\pi d}, \quad p(2) = \frac{a+b-c}{\pi d},$$

see Santaló [5, pp. 77-78].

### 3 Special cases

If  $a = b$ , we have  $k = 1$ ,  $E(1) = 1$ , and therefore

$$p(0) = 1 - \frac{2a(\pi + 2)}{\pi^2 d}, \quad p(1) = \frac{8a}{\pi^2 d}, \quad p(2) = \frac{2a(\pi - 2)}{\pi^2 d}.$$

If  $a \neq 0$  and  $b = 0$ , then  $k = 0$  and  $E(0) = \pi/2$ , and therefore  $P(A \cup B) = P(A) = 2a/(\pi d)$ . This is the result of the classical Buffon needle problem.

### References

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